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On the values of continued fractions: q-series ☆

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Abstract

Using the Poincaré–Perron theorem on the asymptotics of the solutions of linear recurrences it is proved that for a class of q-continued fractions the value of the continued fraction is given by a quotient of the solution and its q-shifted value of the corresponding q-functional equation.

Keywords: q-continued fraction; Recurrence; p-adic

1. Introduction

In the Archimedean case it is known that the values of continued fractions in certain q-continued fraction cases can be given by quotients of q-series. As examples we mention the continued fractions of Rogers–Ramanujan

$$\underset{n=1}{\overset{\infty}{K}}\left(1\left|\begin{matrix}q^n t\\ 1\end{matrix}\right.\right)=\frac{F(t)}{F(qt)},\quad F(t)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q)_n}t^n,\quad |q|<1,\ [20] \tag{1}$$

and

$$\underset{n=1}{\overset{\infty}{K}}\left(1+bqt\left|\begin{matrix}(1+aq^{n+1}t)q^n t\\ 1+bq^{n+1}t\end{matrix}\right.\right)=\frac{H(t)}{H(qt)},\quad |q|<1, \tag{2}$$

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where

$$H(t) = \frac{(qt/a_1)_\infty (qt/a_2)_\infty}{(qt)_\infty (1-t)} \sum_{n=0}^{\infty} \frac{(1-tq^{2n})(t)_n (a_1)_n (a_2)_n q^{n(3n+1)/2} (at^2)^n}{(q)_n (qt/a_1)_n (qt/a_2)_n} \tag{3}$$

and $a = -1/a_1 a_2$, $b = -1/a_1 - 1/a_2$ [3].

Our Theorem 1, which is valid also in the non Archimedean metrics, will find values for a rather wide class of q-continued fractions including the continued fractions (1) and (2) as special cases. However, sometimes our results have very different representations from earlier work, as may seen in the following application of Theorem 1:

$$\mathop{K}_{n=1}^{\infty} \left(1 + bqt \,\middle|\, \begin{matrix} (1+aq^{n+1}t)q^n t \\ 1 + bq^{n+1} t \end{matrix} \right) = \frac{F(t)}{F(qt)}, \quad |q| < 1, \tag{4}$$

where

$$F(t) = (qt/a_2)_\infty \sum_{n=0}^{\infty} \frac{(a_1)_n q^{\binom{n}{2}}}{(qt/a_2)_n (q)_n} \left(-\frac{qt}{a_1} \right)^n \tag{5}$$

and $a = -1/a_1 a_2$, $b = -1/a_1 - 1/a_2$.

2. Notations

The most important class of q-series consist of the q-hypergeometric (basic) series

$${}_k\Phi_l \left(\begin{matrix} a_1, \ldots, a_k \\ b_1, \ldots, b_l \end{matrix} \,\middle|\, q, t \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_k)_n}{(q)_n (b_1)_n \cdots (b_l)_n} t^n,$$

which are defined by using the q-factorials $(a)_0 = (a;q)_0 = 1$ and $(a)_n = (a;q)_n = (1-a)(1-aq)\ldots(1-aq^{n-1})$ for $n \in \mathbb{Z}^+$. We also use $(b,a)_0 = 1$ and $(b,a)_n = (b-a)(b-aq)\ldots(b-aq^{n-1})$ for $n \in \mathbb{Z}^+$. Hence $(a)_n = (1,a)_n$ and especially $(q)_n = (1-q)\ldots(1-q^n)$.

By p we mean an element of the set $\mathbb{P} = \{\infty\} \cup \{p \in \mathbb{Z}^+ \mid p \text{ is a prime}\}$ and we shall use the notation $|\ |_\infty = |\ |$ for the usual absolute value of $\mathbb{C}$ and $|\ |_p$ for the p-adic valuation of the p-adic field $\mathbb{C}_p$, the completion of the algebraic closure of $\mathbb{Q}_p$, normalized by $|p|_p = p^{-1}$.

In the following we shall study convergence of the continued fraction

$$\mathop{K}_{n=1}^{\infty} \left(b_0 \,\middle|\, \begin{matrix} a_n \\ b_n \end{matrix} \right) = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cdots}}, \tag{6}$$

that is, we will determine the limit

$$\lim_{n\to\infty} \frac{A_n}{B_n} \tag{7}$$

of the convergents A_n/B_n, where A_n and B_n satisfy the recurrences

$$A_n = b_nA_{n-1} + a_nA_{n-2}, \quad B_n = b_nB_{n-1} + a_nB_{n-2} \quad \forall n \geqslant 2 \tag{8}$$

with initial values

$$A_0 = b_0, \quad A_1 = b_0b_1 + a_1, \quad B_0 = 1, \quad B_1 = b_1. \tag{9}$$

By the value of the continued fraction (6) we mean the limit (7) when it does exist. Here we note that the continued fraction development using a functional equation, say (11), and starting from $F(t)/F(qt)$ does not necessarily converge to the value $F(t)/F(qt)$, see Perron [18].

3. q-Continued fractions

Through this work we suppose that $|q|_p < 1$ in the given valuation p.

Theorem 1. *Let $s \geqslant 1$, $q, t \in \mathbb{C}_p$, $S_0T_0 \neq 0$, $|S_0|_p \leqslant 1$. Then*

$$\underset{n=1}{\overset{\infty}{K}}\left(T_0 + T_1t + \cdots + T_lt^l \middle| \begin{matrix} t^sq^{s(n-1)}(S_0 + S_1tq^n + \cdots + S_ht^hq^{hn}) \\ T_0 + T_1tq^n + \cdots + T_lt^lq^{ln} \end{matrix} \right)$$
$$= (S_0 + S_1 + \cdots + S_ht^h)\frac{T_0\, G(t)}{S_0\, G(qt)}, \quad |q|_p < 1, \tag{10}$$

where $G : \mathbb{C}_p \to \mathbb{C}_p$ is an analytic function such that $F(t) = t^xG(t)$ is a solution of the functional equation

$$t^sF(q^2t) = -(T_0 + T_1t + \cdots + T_lt^l)F(qt) + (S_0 + S_1t + \cdots + S_ht^h)F(t) \tag{11}$$

satisfying $q^x = S_0/T_0$ and $F(qt) \neq 0$. Moreover, if $h = 0$, then $G : \mathbb{C}_p \to \mathbb{C}_p$ is entire function. The convergence in (10) is uniform with respect to variable t in every bounded subset of C_p.

So the value of the q-fraction (10) is a quotient of power series converging in some disk $|t|_p < r$. Sometimes, such as in Corollary 2, case (20), the value of (18) is given as a quotient of entire functions even $h \neq 0$. This phenomenon occurs when there exist series transformations which give the analytic continuation of $G(t)$ to all of $\mathbb{C}_p$.

The following corollary gives an example of the q-fraction, which value is given directly by a meromorphic function in $\mathbb{C}_p$.

Corollary 1. *Let $a, b, c, q \in \mathbb{C}_p$, then*

$$\underset{n=1}{\overset{\infty}{K}}\left(a + b + c \middle| \begin{matrix} q^n \\ a + bq^n + cq^{2n} \end{matrix} \right) = a\frac{G(t)}{G(qt)}, \tag{12}$$

where

$$G(t) = \sum_{k=0}^{\infty} g_k t^k, \quad g_0 = 1, \quad g_1 = (ab+q)/(qa^2(1-q)), \tag{13}$$

$$g_{k+2} = \frac{q^k}{a^2(1-q^{k+2})}((abq + q^{k+3})g_{k+1} + acg_k) \quad \forall k \in \mathbb{N}. \tag{14}$$

In this work we will not touch explicitly on applications where the polynomials $a_n(q,t)$ and $b_n(q,t)$ are of degree more than two in t. In Corollary 2 the value of a q-continued fraction, where $a_n(q,t)$ is at most second degree and $b_n(q,t)$ is at most first degree polynomial in t, is given as a quotient of q-hypergeometric series. This should be compared to the earlier considerations [1,2,5,7–16,19,20], where frequently the polynomials $a_n(q,t)$ and $b_n(q,t)$ have quite low degrees with respect to t. In [6] a value of continued fraction with degrees 4 and 2 for $a_n(q,t)$ and $b_n(q,t)$, respectively, is given by a quotient of q-hypergeometric series.

Corollary 2. *Let* $A, B, C, D, q \in \mathbb{C}_p$, *then*

$$\operatorname*{K}_{n=1}^{\infty}\left(C + Dt \,\middle|\, \begin{matrix} Aq^n t + Bq^{2n}t^2 \\ C + Dq^n t \end{matrix}\right) = \left(C + \frac{BC}{A}t\right)\frac{G(t)}{G(qt)}, \quad |q|_p < 1, \tag{15}$$

where

$$G(t) = \sum_{n=0}^{\infty} \frac{(CD, -Aq)_n q^{\binom{n}{2}}}{(q)_n}\left(\frac{t}{C^2}\right)^n \quad \text{if } B = 0, \tag{16}$$

$$G(t) = \sum_{n=0}^{\infty} \frac{(\lambda)_n(\gamma)_n}{(q)_n}\left(-\frac{B}{A}t\right)^n \quad \text{if } B \neq 0, \tag{17}$$

and

$$\lambda = \frac{A}{2BC}(D + \sqrt{D^2 + 4Bq}), \quad \gamma = \frac{A}{2BC}(D - \sqrt{D^2 + 4Bq}).$$

Equivalently to Corollary 2 we have

$$\operatorname*{K}_{n=1}^{\infty}\left(C + Dt \,\middle|\, \begin{matrix} Aq^n t + Bq^{2n}t^2 \\ C + Dq^n t \end{matrix}\right) = \left(C + \frac{BC}{A}\lambda t\right)\frac{H(t)}{H(qt)}, \quad |q|_p < 1, \tag{18}$$

where

$$H(t) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-Dt/C)_n(q)_n}\left(\frac{A}{C^2}t\right)^n \quad \text{if } B = 0, \tag{19}$$

$$H(t) = \sum_{n=0}^{\infty} \frac{(\lambda)_n q^{\binom{n}{2}}}{(-B\lambda t/A)_n(q)_n}\left(\frac{\gamma B}{A}t\right)^n \quad \text{if } B \neq 0. \tag{20}$$

Corollary 2 includes the Rogers–Ramanujan continued fraction (1) as the special case $A = C = 1,\ B = D = 0$. The corresponding p-adic result for the Rogers–Ramanujan continued fraction is proved in [16].

The q-continued fraction

$$\mathop{K}_{n=1}^{\infty}\left(1+Dt\,\middle|\,\frac{Aq^n t}{1+Dq^n t}\right) \tag{21}$$

has been extensively studied in the last century. By Corollary 2 the continued fraction (21) has the value

$$\mathop{K}_{n=1}^{\infty}\left(1+Dt\,\middle|\,\frac{Aq^n t}{1+Dq^n t}\right) = \frac{F(t)}{F(qt)} = \frac{G(t)}{G(qt)}, \quad |q|_p < 1, \tag{22}$$

where

$$F(t) = \sum_{n=0}^{\infty} \frac{(-Aq/D)_n q^{\binom{n}{2}}}{(q)_n}(Dt)^n \tag{23}$$

and

$$G(t) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-D)_n (q)_n}(At)^n. \tag{24}$$

The q-fraction (21) appears in Entry 15, Chapter 16 of Ramanujan's second notebook, see [1,7], there the value of the q-fraction (21) in $t = 1$ and $b_0 = 1$ is given by

$$\mathop{K}_{n=1}^{\infty}\left(1\,\middle|\,\frac{Aq^n}{1+Dq^n}\right) = \frac{F(A)}{F(qA)}, \quad F(t) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-Dq)_n (q)_n} t^n. \tag{25}$$

On the other hand, if we replace q by q^2 and put then $D = \beta q$, $A = \alpha$ and $t = 1$ in (22) and (23), then we get the result

$$\mathop{K}_{n=1}^{\infty}\left(1+\beta q\,\middle|\,\frac{\alpha q^{2n}}{1+\beta q^{2n+1}}\right) = \frac{F(1)}{F(q^2)}, \quad |q|_p < 1, \tag{26}$$

where

$$F(t) = \sum_{n=0}^{\infty} \frac{(-\alpha q/\beta; q^2)_n q^{n^2}}{(q^2; q^2)_n}(\beta t)^n, \tag{27}$$

proved by Carlitz [10] and in the special case $\alpha = \beta$ by Gordon [12].

The q-fraction

$$\mathop{K}_{n=1}^{\infty}\left(1+D\,\middle|\,\frac{Aq^n + Bq^{2n}}{1+Dq^n}\right) \tag{28}$$

appears in Ramanujan's lost notebook and is studied in [9] formula $(IV)_R$. The value of (28) may deduced from (18) and (20) and we note that result (4) comes as a special case.

The identity

$$\overset{\infty}{\underset{n=1}{K}}\left(1\middle|\begin{matrix}k+q^n\\1\end{matrix}\right)=\overset{\infty}{\underset{n=1}{K}}\left(\alpha\middle|\begin{matrix}q^n\\\alpha+\beta q^n\end{matrix}\right), \tag{29}$$

where $\alpha=(1+\sqrt{1+4k})/2$ and $\beta=(-1+\sqrt{1+4k})/2$, has been considered by Ramanujan and proved recently in [8]. In identity (29) we note that the second continued fraction falls in the class of continued fractions studied in Corollary 2.

Here we note that modifying our proof of Theorem A we may study also certain q-fractions, where $s=0$. Then it is possible to study the first q-fraction in (29) and also we may study certain q-fractions, which have connections to orthogonal polynomials, see [2].

4. Second order q-functional equations

We shall use the operator $J=J_t$ defined by

$$JF(t)=F(qt)$$

satisfying

$$J(FG)=JFJG$$

whenever the scalar or matrix product of F and G is defined.

Let $F(t)$ satisfy the q-functional equation

$$N(t)F(q^2t)=-A_0(t)F(qt)+B_0(t)F(t) \tag{30}$$

of lowest order, where $N(t), A_0(t), B_0(t)\in\mathbb{C}_p[q,t]$. Equivalently in the matrix form we have

$$NJ\begin{pmatrix}JF\\F\end{pmatrix}=\begin{pmatrix}-A_0 & B_0\\N & 0\end{pmatrix}\begin{pmatrix}JF\\F\end{pmatrix} \tag{31}$$

or

$$NJ\Lambda=\mathscr{P}_0\Lambda, \tag{32}$$

where J operates to the 2-vector Λ and $\mathscr{P}_0$ is the 2×2 matrix. When we denote

$$[F]_n=\prod_{i=0}^{n-1}J^iF,\quad [F]_{-n}=1\ \ \forall n\in\mathbb{N},$$

then we may write

$$[N]_{n+1}J^{n+2}F=(-1)^{n+1}A_nJF+(-1)^nB_nF\quad \forall n\geqslant -2, \tag{33}$$

where $A_{-1}=1$, $B_{-1}=0$ and $A_{-2}=0$, $B_{-2}=1$. Consequently

$$[N]_{n+1}J^{n+1}\Lambda=\mathscr{P}_n\Lambda\quad \forall n\geqslant 0, \tag{34}$$

where

$$\mathscr{P}_n = \begin{pmatrix} (-1)^{n+1}A_n & (-1)^n B_n \\ (-1)^n J^n N A_{n-1} & (-1)^{n-1} J^n N B_{n-1} \end{pmatrix}. \tag{35}$$

Eq. (32) operated by J^{n+1} implies

$$J^{n+1} N J^{n+2} \Lambda = J^{n+1} \mathscr{P}_0 J^{n+1} \Lambda. \tag{36}$$

Multiplication of (36) by $[N]_{n+1}$ and the use of (34) give

$$[N]_{n+2} J^{n+2} \Lambda = J^{n+1} \mathscr{P}_0 \mathscr{P}_n \Lambda. \tag{37}$$

Hence by (34) and (37) we get

Lemma 1.

$$\mathscr{P}_{n+1} = J^{n+1} \mathscr{P}_0 \mathscr{P}_n \quad \forall n \in \mathbb{N}. \tag{38}$$

This fundamental recurrence form implies the following lemmas.

Lemma 2. *The polynomials $A_n(t)$ and $B_n(t)$ satisfy the linear recurrences*

$$A_n(t) = J^n A_0(t) A_{n-1}(t) + J^n B_0(t) J^{n-1} N(t) A_{n-2}(t),$$
$$B_n(t) = J^n A_0(t) B_{n-1}(t) + J^n B_0(t) J^{n-1} N(t) B_{n-2}(t) \quad \forall n \in \mathbb{N} \tag{39}$$

with initial values $A_{-1} = 1$, $B_{-1} = 0$ and $A_{-2} = 0$, $B_{-2} = 1$.

Proof. From formulae (35) and (38) we get

$$\begin{pmatrix} (-1)^{n+1}A_n & (-1)^n B_n \\ (-1)^n J^n N A_{n-1} & (-1)^{n-1} J^n N B_{n-1} \end{pmatrix}$$
$$= \begin{pmatrix} -J^n A_0 & J^n B_0 \\ J^n N & 0 \end{pmatrix} \begin{pmatrix} (-1)^n A_{n-1} & (-1)^{n-1} B_{n-1} \\ (-1)^{n-1} J^{n-1} N A_{n-2} & (-1)^{n-2} J^{n-1} N B_{n-2} \end{pmatrix},$$

which directly gives recurrences (39). □

Lemma 3. *Let $q, t \in \mathbb{C}_p$ satisfy $B_0(q^k t) N(q^k t) \neq 0$ for all $k \in \mathbb{N}$, then*

$$A_n B_{n+1} - A_{n+1} B_n \neq 0 \quad \forall n \in \mathbb{N} \tag{40}$$

and thus the sequences (A_n) and (B_n) are linearly independent solutions of recurrence (39).

Proof. From formula (38) we get

$$\mathscr{P}_n = (J^n \mathscr{P}_0)(J^{n-1} \mathscr{P}_0) \dots (J \mathscr{P}_0) \mathscr{P}_0. \tag{41}$$

Using the determinants

$$\det J^k \mathscr{P}_0 = \det \begin{pmatrix} -J^k A_0 & J^k B_0 \\ J^k N & 0 \end{pmatrix} = -J^k B_0 J^k N$$

we get

$$\det \mathscr{P}_n = (-1)^{n+1} [B_0]_{n+1} [N]_{n+1} \tag{42}$$

and from (37) it follows

$$\det \mathscr{P}_n = J^n N (A_n B_{n-1} - A_{n-1} B_n) \tag{43}$$

giving

$$A_n B_{n-1} - A_{n-1} B_n = (-1)^{n+1} [B_0]_{n+1} [N]_n. \tag{44}$$

Hence (40) follows. □

5. Proof of theorems

Let $F : \mathbb{C}_p \to \mathbb{C}_p$ be a solution the functional equation

$$t^s M(t) F(q^2 t) = -T(t) F(qt) + S(t) F(t), \quad s \in \mathbb{Z}^+ \tag{45}$$

of lowest order, where $M(t), T(t), S(t) \in \mathbb{C}_p[q, t]$. We shall set

$$a_n(t) = t^s q^{s(n-1)} M(q^{n-1} t) S(q^n t), \quad b_n(t) = T(q^n t), \tag{46}$$

where

$$S(t) = \sum_{k=0}^{h} S_k t^k, \quad T(t) = \sum_{k=0}^{l} T_k t^k, \quad M(t) = \sum_{k=0}^{m} M_k t^k.$$

Theorem A. *Let* $s \geqslant 1$, $M_0 S_0 T_0 \neq 0$, $|S_0|_p \leqslant 1$ *and* $q, t \in \mathbb{C}_p$, $|q|_p < 1$. *If* $M(q^k t) S(q^k t) \neq 0$ *for all* $k \in \mathbb{N}$, *then*

$$\mathop{K}_{n=1}^{\infty} \left(b_0(t) \middle| \begin{matrix} a_n(t) \\ b_n(t) \end{matrix} \right) = S(t) \frac{F(t)}{F(qt)}, \tag{47}$$

where $F(t)$ *is a solution of the functional equation* (47), *satisfying* $F(qt) \neq 0$ *and an upper bound condition* $|F(t)|_p \leqslant L_p$ *in a neighbourhood of zero with some* $L_p \in \mathbb{R}^+$.

Proof. First we consider the function

$$H(t) = F(t) \prod_{n=0}^{\infty} S(tq^n),$$

which satisfies the functional equation

$$NH(q^2t) = -A_0H(qt) + B_0H(t),$$
$$A_0(t) = T(t), \quad B_0(t) = 1, \quad N(t) = t^sM(t)S(qt). \tag{48}$$

Using (33) we get

$$R_n = B_n\frac{H(t)}{H(qt)} - A_n = (-1)^n[N]_{n+1}\frac{H(q^{n+2}t)}{H(qt)}. \tag{49}$$

By Lemma 3 there exist an infinity of n such that $B_n(t) \neq 0$ and so we may study the convergence of A_n/B_n towards $H(t)/H(qt)$ by

$$\frac{H(t)}{H(qt)} - \frac{A_n}{B_n} = (-1)^n\frac{t^{s(n+1)}q^{s\binom{n+1}{2}}[M(t)]_{n+1}F(q^{n+2}t)}{B_n(t)F(qt)}. \tag{50}$$

Here $|F(q^{n+2}t)|_p \leqslant L_p$, when n is big enough, say $n \geqslant K_1$, because the upper bound of $F(t)$ near zero. Also there exists an upper bound $L'_p \in \mathbb{N}$ such that

$$|[M(t)]_{n+1}|_p \leqslant L'_p|M_0|_p^{n+1} \quad \forall n \geqslant K_2 \tag{51}$$

because the product

$$M_0^{-n-1}[M(t)]_{n+1} \to M'_p(t) = \prod_{k=0}^{\infty}\frac{M(q^kt)}{M_0},$$

goes to a limit $M'_p \in \mathbb{C}_p$ for every $q, t \in \mathbb{C}_p$ and $|q|_p < 1$.

So in the numerator of (50) the term $q^{s\binom{n+1}{2}}$ determines the convergence while in the denominator we have to study the behaviour of $B_n(t)$.

Lemma 2 gives our starting point which will be the recurrence

$$B_n(t) = T(q^nt)B_{n-1}(t) + t^sq^{s(n-1)}M(q^{n-1}t)S(q^nt)B_{n-2}(t) \tag{52}$$

satisfied by (A_n), (B_n) and (R_n).

(1) *The complex case* $\mathbb{C}$

We shall consider the asymptotic behavior of $B_n(t)$ by using Poincaré–Perron Theorem [17]. First we note that

$$T(q^nt) \to T_0, \quad S(q^nt) \to S_0, \quad M(q^nt) \to M_0. \tag{53}$$

Hence the associated recurrence equation of (52) will be

$$B_n = T_0B_{n-1}, \tag{54}$$

having the characteristic equation

$$Z^2 = T_0Z \tag{55}$$

with roots $\alpha = T_0$ and $\beta = 0$. Here $\alpha \neq \beta$ and thus Poincaré–Perron Theorem [17] gives two linear independent solutions (E_n) and (F_n) of (52) such that

$$\frac{E_{n+1}}{E_n} \to \alpha, \quad \frac{F_{n+1}}{F_n} \to \beta. \tag{56}$$

By (56) we know that for a given $\delta\in\mathbb{R}^+$, $(0<\delta<1)$ there exist $b_1=b_1(\delta)$, $b_2=b_2(\delta)\in\mathbb{R}^+$ and $K_3=K_3(\delta)\in\mathbb{N}$ such that

$$b_1(1-\delta)^n|\alpha|^n\leqslant|E_n|\leqslant b_2(1+\delta)^n|\alpha|^n\quad\forall n\geqslant K_3 \tag{57}$$

and

$$|F_n|\leqslant b_2\delta^n\quad\forall n\geqslant K_3. \tag{58}$$

From (49) we know that for any given $\varepsilon\in\mathbb{R}^+$ there exist $b_3=b_3(\varepsilon)\in\mathbb{R}^+$ and $K_4=K_4(\varepsilon)\in\mathbb{N}$ and such that

$$|R_n|\leqslant(b_3|q|^{sn/2})^n\leqslant\varepsilon^n\quad\forall n\geqslant K_4. \tag{59}$$

If now

$$\frac{B_{n+1}}{B_n}\to\beta \tag{60}$$

then there exist $b_4=b_4(\varepsilon)\in\mathbb{R}^+$ and $K_5=K_5(\varepsilon)\in\mathbb{N}$ such that

$$|B_n|\leqslant b_4\varepsilon^n\quad\forall n\geqslant K_5. \tag{61}$$

By (49)

$$A_n=H'B_n-R_n,\quad H'\in\mathbb{C},$$

and so there exists $b_5=b_5(\varepsilon)\in\mathbb{R}^+$ such that

$$|A_n|\leqslant b_5\varepsilon^n\quad\forall n\geqslant K_6=\max\{K_4,K_5\}. \tag{62}$$

Because (A_n) and (B_n) form a basis for the solution space of recurrence (52) we get to any solution $C_n=aA_n+bB_n$ the upper estimate

$$|C_n|\leqslant b_6\varepsilon^n\quad\forall n\geqslant K_6 \tag{63}$$

for some $b_6=b_6(\varepsilon)\in\mathbb{R}^+$ which clearly contradicts (56). Hence for a given $\delta\in\mathbb{R}^+$, $(0<\delta<1)$ there exist $K=K(\delta)\in\mathbb{N}$ and $b_7=b_7(\delta), b_8=b_8(\delta)\in\mathbb{R}^+$ such that

$$b_7(1-\delta)^n|\alpha|^n\leqslant|B_n|\leqslant b_8(1+\delta)^n|\alpha|^n\quad\forall n\geqslant K \tag{64}$$

and also

$$b_7(1-\delta)^n|\alpha|^n\leqslant|A_n|\leqslant b_8(1+\delta)^n|\alpha|^n\quad\forall n\geqslant K. \tag{65}$$

Taking n enough big in (64), formula (50) implies

$$\left|\frac{H(t)}{H(qt)}-\frac{A_n}{B_n}\right|\leqslant\frac{|t^{s(n+1)}q^{s\binom{n+1}{2}}M_0^{n+1}|L'_\infty L_\infty}{b_7(1-\delta)^n|T_0|^n|F(qt)|}\quad\forall n\geqslant K. \tag{66}$$

(2) *The p-adic case* $\mathbb{C}_p$

Again we shall consider recurrence (52)

$$B_n(t)=T(q^nt)B_{n-1}(t)+t^sq^{s(n-1)}M(q^{n-1}t)S(q^nt)B_{n-2}(t)$$

with K_7 such big that

$$|T(q^nt)B_{n-1}(t)|_p>|t^sq^{s(n-1)}M(q^{n-1}t)S(q^nt)B_{n-2}(t)|_p\quad\forall n\geqslant K_7 \tag{67}$$

giving

$$|B_n(t)|_p = |T(q^n t)B_{n-1}(t)|_p \quad \forall n \geqslant K_7. \tag{68}$$

Hence

$$|B_n(t)|_p \geqslant b_9 |T_0|_p^n \quad \forall n \geqslant K_7. \tag{69}$$

Similarly to (66) we have

$$\left| \frac{H(t)}{H(qt)} - \frac{A_n}{B_n} \right|_p \leqslant \frac{|t^{s(n+1)} q^{s\binom{n+1}{2}} M_0^{n+1}|_p L_p' L_p}{b_9 |T_0|_p^n |F(qt)|_p} \quad \forall n \geqslant K. \tag{70}$$

Together (66) and (70) imply

$$\lim_{n \to \infty} \frac{A_n}{B_n} = \frac{H(t)}{H(qt)} \tag{71}$$

uniformly with respect to variable t in every bounded subset of $\mathbb{C}_p$ for all p finite or infinite.

Let

$$b_n = J^n T(t), \quad a_{n+1} = J^n N(t), \quad \forall n \in \mathbb{N},$$

then (A_n) and (B_n) satisfy the recurrence

$$A_{n+2} = b_{n+2} A_{n+1} + a_{n+2} A_n \quad \forall n \in \mathbb{N} \tag{72}$$

with initial values

$$A_0 = b_0, \quad A_1 = b_0 b_1 + a_1, \quad B_0 = 1, \quad B_1 = b_1. \tag{73}$$

By (72) and (73) we deduce that A_n and B_n are the numerator and the denominator, respectively, of the nth convergent of the continued fraction

$$\underset{n=1}{\overset{\infty}{K}} \left(b_0(t) \middle| \begin{matrix} a_n(t) \\ b_n(t) \end{matrix} \right)$$

for every $n \in \mathbb{N}$. Thus

$$\underset{n=1}{\overset{\infty}{K}} \left(b_0(t) \middle| \begin{matrix} a_n(t) \\ b_n(t) \end{matrix} \right) = \frac{H(t)}{H(qt)}. \tag{74}$$

Hence (74) implies

$$\underset{n=1}{\overset{\infty}{K}} \left(b_0(t) \middle| \begin{matrix} a_n(t) \\ b_n(t) \end{matrix} \right) = S(t) \frac{F(t)}{F(qt)}. \qquad \square \tag{75}$$

To prove Theorem 1 we have to solve the functional equation (11).

Let us define the orbits

$$I(t) = \{tq^n | n \in \mathbb{Z}\}, \quad t \in \mathbb{C}_p,$$

and the index set $\Omega = \{0\} \cup \{t \in \mathbb{C}_p | |q|_p < |t|_p \leqslant 1\}$, which make up a partition of $\mathbb{C}_p$ that is $\coprod_{\omega \in \Omega} I(\omega) = \mathbb{C}_p$. Given $q, t \in \mathbb{C}_p$ and any initial values

$$F(t), F(qt) \in \mathbb{C}_p$$

then the functional equation (11) has unique solution $F(t)$ on the orbit $I(t)$, if $M(q^k t) \neq 0$ for all $k \in \mathbb{Z}$. Thus it should be noted that the functional equation (11) has even non continuous solutions.

However, now we look for the Frobenius series

$$F(t) = \sum_{n=0}^{\infty} g_n t^{n+x}, \quad g_0 = 1, \quad x \in \mathbb{C}_p$$

solution of

$$t^s F(q^2 t) = -(T_0 + \cdots + T_l t^l) F(qt) + (S_0 + \cdots + S_h t^h) F(t). \tag{76}$$

Let E be the shifting operator $E g_n = g_{n+1}$. Readily, the comparison of equal exponents in (76) gives

$$0 = (-T_0 q^x + S_0) g_0 \;\Leftrightarrow\; q^x = S_0 / T_0 \tag{77}$$

and

$$(S(E^{-1}) - q^{n+x} T(q^{-1} E^{-1}) - q^{2n-2s+2x} E^{-s}) g_n = 0 \quad \forall n \in \mathbb{Z}, \tag{78}$$

where we put $g_m = 0$ for all $m \in \mathbb{Z}^-$. Hence

$$\begin{aligned} &(T_0^2 (S_0 + \cdots + S_h E^{-h}) - S_0 T_0 q^n (T_0 + \cdots + T_l q^{-l} E^{-l}) \\ &\quad - S_0^2 q^{2n-2s} E^{-s}) g_n = 0 \quad \forall n \in \mathbb{N}, \end{aligned} \tag{79}$$

which is equivalent to

$$\begin{aligned} &S_0 T_0^2 (1 - q^n) g_n + T_0 (S_1 T_0 - S_0 T_1 q^{n-1}) g_{n-1} + \cdots \\ &\quad + (S_s T_0^2 - S_0 T_0 T_s q^{n-s} - S_0^2 q^{2n-2s}) g_{n-s} + \cdots \\ &\quad + T_0 (S_k T_0 - S_0 T_k q^{n-k}) g_{n-k} = 0 \quad k = \max\{h, l\} \end{aligned} \tag{80}$$

for all $n \in \mathbb{Z}$. The associated recurrence of (80) will be

$$S_0 g_n + S_1 g_{n-1} + \cdots + S_k g_{n-k} = 0 \tag{81}$$

with the characteristic equation

$$S_0 X^h + S_1 X^{h-1} + \cdots + S_h = 0. \tag{82}$$

If $S_h \neq 0$ for some $h \geqslant 1$, then Eq. (82) possess a root system such that $|\alpha_1| \geqslant \cdots \geqslant |\alpha_h| > 0$ and thus Poincaré–Perron Theorem [17] shows that every solution of (80) satisfies

$$|g_n| \leqslant (b_{10} |\alpha_1|)^n \quad \forall n \in \mathbb{N} \tag{83}$$

for some $b_{10} \in \mathbb{R}^+$. Thus the series

$$G(t) = \sum_{n=0}^{\infty} g_n t^n, \quad g_0 = 1,$$

presents an analytic function in some disk $D(0, r) \in \mathbb{C}$ of positive radius r.

The p-adic counterpart goes by elementary estimations.

If $h = 0$, then

$$g_n = \frac{q^{n-l}}{T_0^2(1-q^n)}(T_0T_1q^{l-1}g_{n-1} + \cdots + (T_0T_sq^{l-s} + S_0q^{n+l-2s})g_{n-s} + \cdots + T_0T_lg_{n-l}). \tag{84}$$

So

$$g_n = \frac{q^{[\tau n^2]}}{(q)_n}h_n, \quad \tau = \frac{1}{2\max\{s,l\}+1}, \tag{85}$$

where

$$|h_n|_p \leqslant b_{11}^n \tag{86}$$

for some $b_{11} \in \mathbb{R}^+$. Hence the series $G(t)$ determines an entire function in $\mathbb{C}_p$. □

Proof of Corollary 2. Now we have $s = 1$ and

$$S(t) = Aq + Bqt = S_0 + S_1t,$$
$$T(t) = C + Dt = T_0 + T_1t.$$

Thus $q^x = Aq/C$ and

$$g_k = \frac{A^2q^{2k-1} + ACDq^{k-1} - BC^2}{AC^2(1-q^k)}g_{k-1}, \tag{87}$$

which gives

$$G(t) = \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{n}(A^2q^{2k-1} + ACDq^{k-1} - BC^2)}{(q)_n}\left(\frac{t}{AC^2}\right)^n. \tag{88}$$

By Theorem 2 we get

$$\mathop{K}_{n=1}^{\infty}\left(C + Dt \middle| \begin{matrix} Aq^nt + Bq^{2n}t^2 \\ C + Dq^nt \end{matrix}\right) = (Aq + Bqt)\frac{G(t)}{q^xG(qt)},$$

which is (15) with (16) when $B = 0$. If $B \neq 0$, then we factor the numerator in (88) to get

$$G(t) = \sum_{n=0}^{\infty} \frac{(\gamma)_n(\lambda)_n}{(q)_n}\left(\frac{-Bt}{A}\right)^n,$$

which is (17), where

$$\lambda = \frac{A}{2BC}(D + \sqrt{D^2 + 4Bq}), \quad \gamma = \frac{A}{2BC}(D - \sqrt{D^2 + 4Bq}).$$

Finally using Heine's transformation

$$
{}_2\Phi_1\left(\begin{matrix} a, b \\ c \end{matrix}\middle| x\right) = \frac{(c/b)_\infty (bx)_\infty}{(c)_\infty (x)_\infty} {}_2\Phi_1\left(\begin{matrix} abx/c, b \\ bx \end{matrix}\middle| \frac{c}{b}\right) \tag{89}
$$

(see [4, 10.10.1]) we have

$$
\begin{aligned}
G(t) &= \frac{(-B\lambda t/A)_\infty}{(-Bt/A)_\infty} H(t) \\
&= \frac{(-B\lambda t/A)_\infty}{(-Bt/A)_\infty} \sum_{n=0}^{\infty} \frac{(\lambda)_n q^{\binom{n}{2}}}{(-B\lambda t/A)_n (q)_n} \left(\frac{\gamma B}{A} t\right)^n.
\end{aligned} \tag{90}
$$

Identity (90) proves (18) with (20), which reduces to (19), when $B = 0$.

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